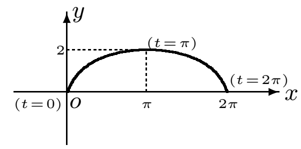
積分の応用 ≪基礎≫ 小テスト　解答例　($No.6$)

問 曲線 C : $x = t - \sin t$, $y = 1 - \cos t$ $(0 \leqq t \leqq 2\pi)$ と x 軸で囲まれた図形を A とする。このとき次の各問いに答えよ。

1. 図形 A の面積を求めよ。



（解）$\dfrac{dx}{dt} = 1 - \cos t$

$$S = 2 \cdot \int_0^{\pi} \left| y \frac{dx}{dt} \right| dt = 2 \int_0^{\pi} |(1-\cos t)(1-\cos t)|\, dt = 2 \int_0^{\pi} (1-\cos t)^2 dt$$

$$= 2 \int_0^{\pi} \left(2\sin^2 \frac{t}{2} \right)^2 dt = 8 \int_0^{\pi} \sin^4 \frac{t}{2} dt \quad \Longleftarrow \quad \boxed{\text{半角の公式より } \sin^2 \tfrac{t}{2} = \tfrac{1-\cos t}{2} \ \therefore 1 - \cos t = 2\sin^2 \tfrac{t}{2}}$$

$\theta = \dfrac{1}{2}t$ とおくと　$\dfrac{d\theta}{dt} = \dfrac{1}{2}$　$\therefore 2d\theta = dt$

t	$0 \to \pi$
θ	$0 \to \frac{\pi}{2}$

$$S = 8 \int_0^{\frac{\pi}{2}} \sin^4 \theta \cdot 2d\theta = 16 \int_0^{\frac{\pi}{2}} \sin^4 \theta d\theta = 16 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 3\pi \quad 〃$$

2. 図形 A を x 軸のまわりに回転してできる回転体の体積を求めよ。

（解）$V = 2 \cdot \pi \displaystyle\int_0^{\pi} y^2 \left| \frac{dx}{dt} \right| dt = 2\pi \int_0^{\pi} (1-\cos t)^2 (1-\cos t) dt = 2\pi \int_0^{\pi} (1-\cos t)^3 dt$

$$= 2\pi \int_0^{\pi} \left(2\sin^2 \frac{t}{2} \right)^3 dt = 16\pi \int_0^{\pi} \sin^6 \frac{t}{2} dt$$

$\theta = \dfrac{1}{2}t$ とおくと　$\dfrac{d\theta}{dt} = \dfrac{1}{2}$　$\therefore 2d\theta = dt$

t	$0 \to \pi$
θ	$0 \to \frac{\pi}{2}$

$$V = 16\pi \int_0^{\frac{\pi}{2}} \sin^6 \theta \cdot 2\, d\theta = 32\pi \int_0^{\frac{\pi}{2}} \sin^6 \theta\, d\theta = 32\pi \cdot \frac{5}{\not{6}_2} \cdot \frac{\not{3}^1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 32\pi \cdot \frac{5}{32}\pi = 5\pi^2 \quad 〃$$

3. 曲線 C の長さを求めよ。

（解）$\dfrac{dx}{dt} = 1 - \cos t$, $\dfrac{dy}{dt} = -(-\sin t) = \sin t$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2(1-\cos t)}$$

$$= \sqrt{2 \cdot 2\sin^2 \frac{t}{2}} = \sqrt{\left(2\sin\frac{t}{2}\right)^2} = \left| 2\sin\frac{t}{2} \right| = 2\sin\frac{t}{2} \qquad \left(\because\ 0 \leqq \frac{t}{2} \leqq \pi \text{ より } \sin\frac{t}{2} \geqq 0 \text{ だから} \right)$$

$$L = 2 \cdot \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 2 \int_0^{\pi} 2\sin\frac{t}{2} dt = 4 \int_0^{\pi} \sin\frac{t}{2} dt$$

$\theta = \dfrac{1}{2}t$ とおくと　$\dfrac{d\theta}{dt} = \dfrac{1}{2}$　$\therefore 2d\theta = dt$

t	$0 \to \pi$
θ	$0 \to \frac{\pi}{2}$

$$L = 4\int_0^{\frac{\pi}{2}} \sin\theta \cdot 2d\theta = 8\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = 8\Big[-\cos\theta \Big]_0^{\pi} ! - 8\Big[\cos\theta \Big]_0^{\pi} = -8\left(\cos\frac{\pi}{2} - \cos 0 \right) = 8 \quad 〃$$

4. 曲線 C を x 軸のまわりに回転してできる回転面の面積を求めよ。

（解）$S = 2 \cdot 2\pi \displaystyle\int_0^{\pi} |y| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 4\pi \int_0^{\pi} |1-\cos t| \cdot 2\sin\frac{t}{2} dt$

$$= 4\pi \int_0^{\pi} 2\sin^2\frac{t}{2} \cdot 2\sin\frac{t}{2} dt = 16\pi \int_0^{\pi} \sin^3\frac{t}{2} dt$$

$\theta = \dfrac{1}{2}t$ とおくと　$\dfrac{d\theta}{dt} = \dfrac{1}{2}$　$\therefore 2d\theta = dt$

t	$0 \to \pi$
θ	$0 \to \frac{\pi}{2}$

$$S = 16\pi \int_0^{\frac{\pi}{2}} \sin^3\theta \cdot 2d\theta = 32\pi \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta = 32\pi \cdot \frac{2}{3} \cdot 1 = \frac{64}{3}\pi \quad 〃$$