積分法 ≪基礎≫ 小テスト　$(No.11)$　解答例

1. 次の定積分の値を求めよ。

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^9 x dx$

（解） $\displaystyle\int_0^{\frac{\pi}{2}} \sin^9 x dx = \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{128}{315}$ 〃

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^8 x dx$

（解） $\displaystyle\int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35}{256}\pi$ 〃

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x dx$

（解） $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x dx = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^3 x dx = \int_0^{\frac{\pi}{2}} \cos^3 x dx - \int_0^{\frac{\pi}{2}} \cos^5 x dx$

$\displaystyle = \frac{2}{3}\cdot 1 - \frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{2}{3} - \frac{8}{15} = \frac{10}{15} - \frac{8}{15} = \frac{2}{15}$ 〃

(4) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^4 3x dx$

（解）$3x = t$ とおくと、$\displaystyle\frac{dt}{dx} = 3$ $\therefore$ $\displaystyle\frac{1}{3}dt = dx$

x	0	$\to$	$\frac{\pi}{6}$
t	0	$\to$	$\frac{\pi}{2}$

$\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^4 3x dx = 2\int_0^{\frac{\pi}{6}} \sin^4 3x dx = 2\int_0^{\frac{\pi}{2}} \sin^4 t \cdot \frac{1}{3} dt$

$\displaystyle = \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin^4 t dt = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{8}$ 〃

≪考え方≫

$f(x) = \sin^4 3x$ 　とおくと

$f(-x) = \sin^4(-3x)$

$= (-\sin 3x)^4 = \sin^4 3x$

$\therefore f(x) = f(-x)$

よって、

$f(x)$ は偶関数である。

2. 次の不定積分を求めよ。

(1) $\displaystyle\int \sin 4x \sin 3x dx$

（解） 与式 $\displaystyle = -\frac{1}{2}\int \{\cos(4x+3x) - \cos(4x-3x)\}dx$

$\displaystyle = -\frac{1}{2}\int(\cos 7x - \cos x)dx = -\frac{1}{2}\left(\frac{1}{7}\sin 7x - \sin x\right)$ 〃

≪積を和に直す公式の求め方≫

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

$\therefore \cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$

$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$

$\therefore \sin\alpha\sin\beta = -\frac{1}{2}\{\cos(\alpha+\beta) - \cos(\alpha-\beta)\}$

(2) $\displaystyle\int \sec x dx$ 　$\left(公式 \displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| を利用\right)$

（解） $\displaystyle\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{1}{1-\sin^2 x}\cdot\cos x dx$

$\sin x = t$ とおくと、$\displaystyle\frac{dt}{dx} = \cos x$ 　$\therefore dt = \cos x dx$ 　また $-1 \leqq \sin x \leqq 1$ であるから

与式 $\displaystyle = \int \frac{1}{1-t^2} dt = -\int \frac{1}{t^2-1^2} dt = -\frac{1}{2\cdot 1}\log\left|\frac{t-1}{t+1}\right| = -\frac{1}{2}\log\left|\frac{\sin x - 1}{\sin x + 1}\right|$

$\displaystyle = -\frac{1}{2}\log\left(\frac{1-\sin x}{1+\sin x}\right) = \frac{1}{2}\log\left(\frac{1-\sin x}{1+\sin x}\right)^{-1} = \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)$ 〃

（別解）公式を利用しない場合　《上の途中から》

与式 $\displaystyle = \int \frac{1}{1-t^2} dt = -\int \frac{1}{t^2-1^2} dt = -\int \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$

$\displaystyle = -\frac{1}{2}\int \left\{\frac{(t-1)'}{t-1} - \frac{(t+1)'}{t+1}\right\} dt = -\frac{1}{2}\left(\log|t-1| - \log|t+1|\right) = -\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$

$\displaystyle = -\frac{1}{2}\log\left|\frac{\sin x - 1}{\sin x + 1}\right| = -\frac{1}{2}\log\left(\frac{1-\sin x}{1+\sin x}\right) = \frac{1}{2}\log\left(\frac{1-\sin x}{1+\sin x}\right)^{-1} = \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)$ 〃